

Jacobi zeta function and action-angle coordinates for the pendulum

Alain J. Brizard

Department of Chemistry and Physics, Saint Michael's College, Colchester, VT 05439, USA

The action-angle coordinates for the planar pendulum problem are expressed in terms of the Jacobi elliptic functions and integrals. In particular, we show that the Jacobi zeta function generates the canonical transformation from the pendulum coordinates ϑ and $p \equiv \partial\vartheta/\partial t$ to the action-angle coordinates (J, ζ) for both the librating pendulum and the rotating pendulum.

I. INTRODUCTION

It is often said that, in order to understand the Jacobi elliptic functions, one only needs to study the pendulum problem [1]. Jacobi elliptic functions are doubly-periodic functions that are useful in solving many problems in classical mechanics, general relativity and cosmology, as well as the soliton solutions of several nonlinear partial differential equations [2–6]. For an excellent introduction to the Jacobi elliptic functions see Ref. [2] while a survey of the properties of elliptic functions and integrals is given in Refs. [7, 8]. The Jacobi functions are familiar to many physicists because they transform into the well-known singly-periodic trigonometric (or hyperbolic trigonometric) functions in the limit where one of the two Jacobi periods becomes infinite.

In the present paper, we show that another Jacobi elliptic function, the Jacobi zeta function [2], plays a fundamental role in generating the canonical transformation from the pendulum coordinates ϑ and $p \equiv \partial\vartheta/\partial t$ to the action-angle coordinates (J, ζ) for both the librating and rotating motions of the pendulum.

We begin with a brief review of the solutions of librating and rotating motion of the pendulum problem expressed in terms of the Jacobi elliptic functions [2–5]. The solution for the problem of the libration motion of a pendulum is expressed in terms of the Jacobi elliptic function $\text{sn}(t|\kappa)$ [9] as

$$\vartheta_\ell(t, \kappa) = 2 \arcsin[\sqrt{\kappa} \text{sn}(t|\kappa)], \quad (1)$$

where t denotes the dimensionless time and the Jacobi parameter κ is used to define the dimensionless energy of the librating pendulum $\epsilon \equiv 2\kappa < 2$. In the low-energy (small-amplitude) limit $\kappa \ll 1$, we recover the simple-pendulum solution $\vartheta_\ell(t, \kappa) \simeq 2\sqrt{\kappa} \sin t$ from the librating solution (1), where $\text{sn}(t|\kappa) \simeq \sin t$.

For the rotation motion of the pendulum ($\epsilon > 2$), the Jacobi elliptic function $\text{sn}(t|\kappa)$ in Eq. (1) can be evaluated according to the relation ($\kappa > 1$)

$$\sqrt{\kappa} \text{sn}(t|\kappa) \equiv \text{sn}(\sqrt{\kappa} t|\kappa^{-1}), \quad (2)$$

and Eq. (1) is replaced with

$$\vartheta_r(t, \kappa) = 2 \arcsin[\text{sn}(\sqrt{\kappa} t|\kappa^{-1})]. \quad (3)$$

Note that the librating pendulum oscillates between the turning points $\pm \vartheta_{\ell 0}(\kappa)$, where

$$0 \leq \vartheta_{\ell 0}(\kappa) \equiv 2 \arcsin(\sqrt{\kappa}) < \pi, \quad (4)$$

while the range of motion for the rotating pendulum is $-\pi \leq \vartheta_r \leq \pi$ (where $-\pi$ and π are now considered to be identical points).

The canonical momenta $p \equiv \partial\vartheta/\partial t$ for the libration solution (1) and the rotation solution (3) are expressed as

$$p_\ell(t, \kappa) = 2\sqrt{\kappa} \text{cn}(t|\kappa), \quad (5)$$

$$p_r(t, \kappa) = 2\sqrt{\kappa} \text{dn}(\sqrt{\kappa} t|\kappa^{-1}), \quad (6)$$

where we used the relations ($\kappa > 1$)

$$\left. \begin{aligned} \text{cn}(t|\kappa) &\equiv \text{dn}(\sqrt{\kappa} t|\kappa^{-1}) \\ \text{dn}(t|\kappa) &\equiv \text{cn}(\sqrt{\kappa} t|\kappa^{-1}) \end{aligned} \right\}. \quad (7)$$

Using these solutions, and the trigonometric identity $1 - \cos \vartheta = 2 \sin^2(\vartheta/2)$, we note that the energy equation

$$\frac{1}{2} p^2 + (1 - \cos \vartheta) = 2\kappa \equiv \epsilon \quad (8)$$

follows from the Jacobi identities

$$\left. \begin{aligned} \text{cn}^2(t|\kappa) + \text{sn}^2(t|\kappa) \\ \text{dn}^2(\sqrt{\kappa} t|\kappa^{-1}) + \kappa^{-1} \text{sn}^2(\sqrt{\kappa} t|\kappa^{-1}) \end{aligned} \right\} = 1 \quad (9)$$

for the librating and rotating solutions, respectively.

Lastly, the separatrix solution (which separates the librating solution from the rotating solution) is obtained by substituting the value $\kappa = 1$ in Eqs. (1) and (5) [or Eqs. (3) and (6)]:

$$\left. \begin{aligned} \vartheta_s(t) &= 2 \arcsin(\tanh t) \\ p_s(t) &= 2 \text{sech } t \end{aligned} \right\}, \quad (10)$$

where we used the identities $\text{sn}(t|1) = \tanh t$ and $\text{cn}(t|1) = \text{dn}(t|1) = \text{sech } t$. Note that the turning points $\pm \pi$ of the separatrix solution (10) are now reached only after an infinite period of time.

The remainder of this paper is organized as follows. In Sec. II, we introduce the action-angle coordinates (J, ζ)

for each type of pendulum motion, and the librating and rotating solutions are reformulated in terms of Jacobi elliptic functions of (J, ζ) . In Sec. III, we show that the transformation $(p, \vartheta) \rightarrow (J, \zeta)$ is a canonical transformation and we find the generating function S for this transformation for each type of motion in Sec. IV. Here, we show that the Jacobi zeta function appears naturally as the generating function for the canonical transformation $(p, \vartheta) \rightarrow (J, \zeta)$ for both types of motion.

II. ACTION-ANGLE COORDINATES

In this Section, we calculate explicit expressions for the action-angle coordinates associated with the pendulum problem discussed in Sec. I. We begin with the action coordinate

$$J \equiv \frac{1}{2\pi} \oint p d\vartheta \quad (11)$$

for the librating and rotating motions of the pendulum, where the magnitude of the momentum

$$|p|(\vartheta, \kappa) = 2 \sqrt{\kappa - \sin^2(\vartheta/2)} \quad (12)$$

vanishes at $\vartheta = \pm \vartheta_{\ell 0}(\kappa)$ for the librating case ($\kappa < 1$) while the motion of a co-rotating pendulum ($p > 0$) is separated from that of a counter-rotating pendulum ($p < 0$).

First, the action coordinate for the librating pendulum is calculated from Eq. (11) as

$$\begin{aligned} J_{\ell}(\kappa) &= \frac{4\kappa}{\pi} \int_{-\pi/2}^{\pi/2} \frac{\cos^2 \phi d\phi}{\sqrt{1 - \kappa \sin^2 \phi}} \\ &= \frac{8}{\pi} \left[E(\kappa) - (1 - \kappa) K(\kappa) \right], \end{aligned} \quad (13)$$

where we used the substitution $\sin(\vartheta/2) = \sqrt{\kappa} \sin \phi$ in Eq. (12), while $K(\kappa)$ and $E(\kappa)$ denote the complete elliptic integrals of the first and second kinds [7]. In the low-energy limit ($\kappa \ll 1$), Eq. (13) yields $J_{\ell}(\kappa) \simeq 2\kappa$.

Next, the action coordinate for the co-rotating pendulum is calculated from Eq. (11) as

$$\begin{aligned} J_r(\kappa) &= \frac{2\sqrt{\kappa}}{\pi} \int_{-\pi/2}^{\pi/2} \sqrt{1 - \kappa^{-1} \sin^2 \phi} d\phi \\ &= \frac{4\sqrt{\kappa}}{\pi} E(\kappa^{-1}), \end{aligned} \quad (14)$$

where we used the substitution $\vartheta = 2\phi$ in Eq. (12); the action of a counter-rotating pendulum ($p < 0$) has the value $-J_r(\kappa)$ at the same energy $\epsilon = 2\kappa$. Note that the action coordinate is discontinuous at the separatrix ($\kappa = 1$): $J_{\ell}(1) = 8/\pi$ and $J_r(1) = 4/\pi$. The extra factor of 2 comes from the fact that the full librating cycle requires a passage from $-\vartheta_{\ell 0}(\kappa)$ to $\vartheta_{\ell 0}(\kappa)$ and back to $-\vartheta_{\ell 0}(\kappa)$, while the full rotating cycle only requires a one-way passage from $-\pi$ to π (which is identical to $-\pi$).

Once the action coordinates $J(\epsilon)$ have been calculated as functions of energy $\epsilon = 2\kappa$, we can calculate the pendulum frequencies $\omega(\epsilon) \equiv (\partial J / \partial \epsilon)^{-1}$. The librating-pendulum frequency is defined from Eq. (13) as

$$\omega_{\ell}(\kappa) \equiv \left(\frac{\partial J_{\ell}}{\partial \epsilon} \right)^{-1} = \frac{\pi}{2K(\kappa)}, \quad (15)$$

where we used the expression

$$\frac{d}{d\kappa} \left[E(\kappa) - (1 - \kappa) K(\kappa) \right] = \frac{1}{2} K(\kappa).$$

In the low-energy limit ($\kappa \ll 1$), Eq. (15) yields $\omega_{\ell} \simeq 1$ (i.e., $T_{\ell} \simeq 2\pi$). The rotating-pendulum frequency, on the other hand, is defined from Eq. (14) as

$$\omega_r(\kappa) \equiv \left(\frac{\partial J_r}{\partial \epsilon} \right)^{-1} = \frac{\pi \sqrt{\kappa}}{K(\kappa^{-1})}, \quad (16)$$

where we used the expression

$$\frac{d}{d\kappa} \left[\sqrt{\kappa} E(\kappa^{-1}) \right] = \frac{K(\kappa^{-1})}{2\sqrt{\kappa}}.$$

By comparing Eqs. (13)-(14) and Eqs. (15)-(16), we obtain the relations

$$K(\kappa^{-1}) = \sqrt{\kappa} K(\kappa), \quad (17)$$

$$\sqrt{\kappa} E(\kappa^{-1}) = E(\kappa) - (1 - \kappa) K(\kappa). \quad (18)$$

Hence, the pendulum problem naturally gives the extension of the complete elliptic integrals E and K for $\kappa > 1$. As discussed with the separatrix solution (10), the frequencies (15)-(16) go to zero (i.e., the periods become infinite) in the limit $\kappa \rightarrow 1$ (since K goes to infinity in that limit).

Lastly, the librating solutions (1) and (5) can be re-expressed in terms of the action-angle (J_{ℓ}, ζ_{ℓ}) coordinates as

$$\vartheta_{\ell}(J_{\ell}, \zeta_{\ell}) = 2 \arcsin \left[\sqrt{\kappa} \operatorname{sn} \left(\frac{2K}{\pi} \zeta_{\ell} \mid \kappa \right) \right], \quad (19)$$

$$p_{\ell}(J_{\ell}, \zeta_{\ell}) = 2\sqrt{\kappa} \operatorname{cn} \left(\frac{2K}{\pi} \zeta_{\ell} \mid \kappa \right), \quad (20)$$

where the libration angle ζ_{ℓ} is defined as

$$\zeta_{\ell} \equiv \omega_{\ell} t = (\pi/2) t / K(\kappa), \quad (21)$$

and $\kappa(J_{\ell})$ is obtained from Eq. (13). In the low-energy limit ($\kappa \ll 1$), Eqs. (19)-(20) become

$$\left. \begin{aligned} \vartheta_{\ell}(J_{\ell}, \zeta_{\ell}) &\simeq 2\sqrt{\kappa} \sin \zeta_{\ell} \\ p_{\ell}(J_{\ell}, \zeta_{\ell}) &\simeq 2\sqrt{\kappa} \cos \zeta_{\ell} \end{aligned} \right\}, \quad (22)$$

where $J_{\ell} \simeq 2\kappa$ and $\zeta_{\ell} \simeq t$. The rotating solutions (3) and (6), on the other hand, can also be re-expressed in

terms of the action-angle (J_r, ζ_r) coordinates as

$$\vartheta_r(J_r, \zeta_r) = 2 \arcsin \left[\operatorname{sn} \left(K(\kappa^{-1}) \frac{\zeta_r}{\pi} \mid \kappa^{-1} \right) \right], \quad (23)$$

$$p_r(J_r, \zeta_r) = 2\sqrt{\kappa} \operatorname{dn} \left(K(\kappa^{-1}) \frac{\zeta_r}{\pi} \mid \kappa^{-1} \right), \quad (24)$$

where the rotation angle ζ_r is defined as

$$\zeta_r \equiv \omega_r t = \pi \sqrt{\kappa} t / K(\kappa^{-1}), \quad (25)$$

and $\kappa(J_r)$ is obtained from Eq. (14).

III. CANONICAL TRANSFORMATION

We now show that the phase-space transformation

$$(\vartheta, p) \rightarrow (\zeta, J) \quad (26)$$

is canonical for both the librating solution [Eqs. (19)-(20)] and the rotating solution [Eqs. (23)-(24)] by proving the canonical relation

$$\frac{\partial \vartheta}{\partial \zeta} \frac{\partial p}{\partial J} - \frac{\partial \vartheta}{\partial J} \frac{\partial p}{\partial \zeta} = 1 \quad (27)$$

for each solution.

A. Libration Case

We first consider the case of the librating pendulum ($\kappa < 1$). For this case (where $t = \zeta/\omega_\ell$), the partial derivatives in Eq. (27) are

$$\left. \frac{\partial \vartheta}{\partial \zeta} \right|_J \equiv \frac{1}{\omega_\ell} \frac{\partial \vartheta}{\partial t}, \quad (28)$$

$$\left. \frac{\partial \vartheta}{\partial J} \right|_\zeta \equiv \frac{\omega_\ell}{2} \left(\frac{\partial}{\partial \kappa} + \frac{1}{\Omega_\ell} \frac{\partial}{\partial t} \right), \quad (29)$$

where $\partial/\partial t$ and $\partial/\partial \kappa$ are understood to be at constant κ and t , respectively, and we have introduced the definition

$$\frac{1}{\Omega_\ell} \equiv \left. \frac{\partial t}{\partial \kappa} \right|_\zeta = \frac{t}{2\kappa(1-\kappa)} \left[\frac{E}{K} - (1-\kappa) \right]. \quad (30)$$

We thus use Eqs. (19)-(20) and (28) to obtain

$$\frac{\partial \vartheta}{\partial \zeta} = \frac{1}{\omega_\ell} \frac{\partial \vartheta}{\partial t} = \frac{2\sqrt{\kappa}}{\omega_\ell} \operatorname{cn}, \quad (31)$$

$$\frac{\partial p}{\partial \zeta} = \frac{1}{\omega_\ell} \frac{\partial p}{\partial t} = -\frac{2\sqrt{\kappa}}{\omega_\ell} \operatorname{sn} \operatorname{dn}, \quad (32)$$

and then, using Eq. (29), we obtain

$$\frac{\partial \vartheta}{\partial J} = \frac{\omega_\ell}{2\sqrt{\kappa}} \left[\operatorname{sd} + \frac{2\kappa}{\operatorname{dn}} \left(\frac{\partial \operatorname{sn}}{\partial \kappa} + \frac{1}{\Omega_\ell} \frac{\partial \operatorname{sn}}{\partial t} \right) \right], \quad (33)$$

$$\frac{\partial p}{\partial J} = \frac{\omega_\ell}{2\sqrt{\kappa}} \left[\operatorname{cn} + 2\kappa \left(\frac{\partial \operatorname{cn}}{\partial \kappa} + \frac{1}{\Omega_\ell} \frac{\partial \operatorname{cn}}{\partial t} \right) \right], \quad (34)$$

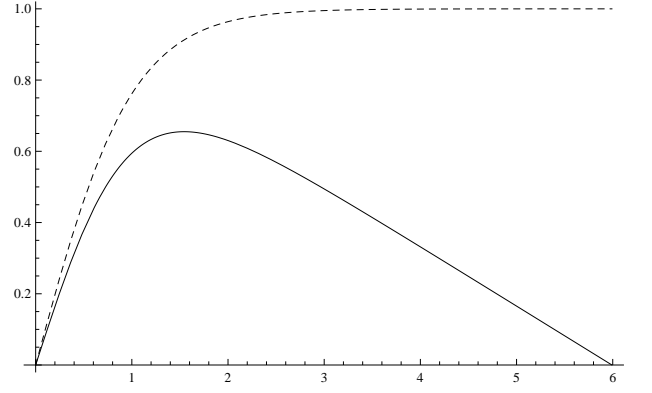


FIG. 1: Plot of $Z(t|\kappa)$ versus $0 \leq t \leq K(\kappa)$ for $\kappa = 0.9999$ (solid) and $Z(t|1) = \tanh t$ (dashed).

where $\operatorname{sd} \equiv \operatorname{sn}/\operatorname{dn}$ (following standard notation for Jacobi elliptic functions). By inserting these expressions into Eq. (27), we obtain

$$\frac{\partial \vartheta}{\partial \zeta} \frac{\partial p}{\partial J} - \frac{\partial \vartheta}{\partial J} \frac{\partial p}{\partial \zeta} = \frac{\partial}{\partial \kappa} \left[\kappa (\operatorname{cn}^2 + \operatorname{sn}^2) \right] = 1, \quad (35)$$

which proves the canonical relation (27) for the librating case when the identity (9) is used.

In writing Eqs. (33)-(34) explicitly, the partial derivatives

$$\frac{\partial \operatorname{sn}}{\partial \kappa} + \frac{1}{\Omega_\ell} \frac{\partial \operatorname{sn}}{\partial t} = \frac{\kappa \operatorname{sn} \operatorname{cn}^2 - Z \operatorname{cn} \operatorname{dn}}{2\kappa(1-\kappa)}, \quad (36)$$

$$\frac{\partial \operatorname{cn}}{\partial \kappa} + \frac{1}{\Omega_\ell} \frac{\partial \operatorname{cn}}{\partial t} = \frac{Z \operatorname{sn} \operatorname{dn} - \kappa \operatorname{cn} \operatorname{sn}^2}{2\kappa(1-\kappa)}, \quad (37)$$

are expressed in terms of the Jacobi zeta function [2]

$$Z(t|\kappa) \equiv \int_0^t \left(\operatorname{dn}^2(s|\kappa) - \frac{E}{K} \right) ds. \quad (38)$$

The Jacobi zeta function (38) has odd parity: $Z(-t|\kappa) = -Z(t|\kappa)$, it has a period of $2K(\kappa)$: $Z(t+2K|\kappa) = Z(t|\kappa)$, and it vanishes at $t = nK$, with $n = 0, \pm 1, \pm 2, \dots$: $Z(nK|\kappa) = 0$. The Jacobi zeta function (38) is also evaluated for the separatrix case ($\kappa = 1$) as

$$Z(t|1) = \int_0^t \operatorname{sech}^2 s \, ds = \tanh t. \quad (39)$$

Figure 1 shows the plot of $Z(t|\kappa)$ for $\kappa = 0.9999$ (solid) and the plot of $Z(t|1) = \tanh t$ (dashed). In the low-energy limit ($\kappa \ll 1$), we use $\operatorname{dn}^2(s|\kappa) - E/K \simeq \frac{1}{2} \kappa \cos 2s$ in Eq. (38), which yields $Z(t|\kappa) \simeq \frac{1}{4} \kappa \sin 2t$.

Lastly, using Eqs. (36)-(37), Eqs. (33)-(34) become

$$\frac{\partial \vartheta}{\partial J} = \frac{\omega_\ell}{2\sqrt{\kappa}(1-\kappa)} \left(\operatorname{sn} \operatorname{dn} - Z \operatorname{cn} \right), \quad (40)$$

$$\frac{\partial p}{\partial J} = \frac{\omega_\ell}{2\sqrt{\kappa}(1-\kappa)} \left[\operatorname{cn} (\operatorname{dn}^2 - \kappa) + Z \operatorname{sn} \operatorname{dn} \right], \quad (41)$$

and we verify again that the canonical relation (27) is satisfied.

B. Rotation Case

Next, we consider the case of the rotating pendulum ($\kappa > 1$). For this case (where $u \equiv \sqrt{\kappa} t = \sqrt{\kappa} \zeta / \omega_r$), the partial derivatives in Eq. (27) are

$$\left. \frac{\partial}{\partial \zeta} \right|_J \equiv \frac{\sqrt{\kappa}}{\omega_r} \frac{\partial}{\partial u}, \quad (42)$$

$$\left. \frac{\partial}{\partial J} \right|_\zeta \equiv \frac{\omega_r}{2} \left(\frac{\partial}{\partial \kappa} + \frac{1}{\Omega_r} \frac{\partial}{\partial u} \right), \quad (43)$$

where $\partial/\partial u$ and $\partial/\partial \kappa$ are understood to be at constant κ and u , respectively, and we have introduced the definition

$$\frac{1}{\Omega_r} \equiv \left. \frac{\partial u}{\partial \kappa} \right|_\zeta = -\frac{u}{2(\kappa - 1)} \left[\frac{E(\kappa^{-1})}{K(\kappa^{-1})} - (1 - \kappa^{-1}) \right]. \quad (44)$$

We thus use Eqs. (23)-(24) and (42) to obtain

$$\frac{\partial \vartheta}{\partial \zeta} = \frac{2\sqrt{\kappa}}{\omega_r} \overline{\text{dn}}, \quad (45)$$

$$\frac{\partial p}{\partial \zeta} = -\frac{2}{\omega_r} \overline{\text{sn cn}}, \quad (46)$$

where we use the overbar notation $\overline{pq} \equiv pq(u|\kappa^{-1})$, and then, using Eqs. (23)-(24), we obtain

$$\frac{\partial \vartheta}{\partial J} = \frac{\omega_r}{\overline{\text{cn}}} \left(\frac{\partial \overline{\text{sn}}}{\partial \kappa} + \frac{1}{\Omega_r} \frac{\partial \overline{\text{sn}}}{\partial u} \right), \quad (47)$$

$$\frac{\partial p}{\partial J} = \frac{\omega_r}{2\sqrt{\kappa}} \left[\overline{\text{dn}} + 2\kappa \left(\frac{\partial \overline{\text{dn}}}{\partial \kappa} + \frac{1}{\Omega_r} \frac{\partial \overline{\text{dn}}}{\partial u} \right) \right]. \quad (48)$$

By inserting these expressions into Eq. (27), we obtain

$$\frac{\partial \vartheta}{\partial \zeta} \frac{\partial p}{\partial J} - \frac{\partial \vartheta}{\partial J} \frac{\partial p}{\partial \zeta} = \frac{\partial}{\partial \kappa} \left[\kappa \left(\overline{\text{dn}}^2 + \kappa^{-1} \overline{\text{sn}}^2 \right) \right] = 1, \quad (49)$$

which proves the canonical relation (27) for the rotating case when the identity (9) is used.

In writing Eqs. (47)-(48), the partial derivatives

$$\frac{\partial \overline{\text{sn}}}{\partial \kappa} + \frac{1}{\Omega_r} \frac{\partial \overline{\text{sn}}}{\partial u} = \frac{\overline{Z} \overline{\text{cn}} \overline{\text{dn}} - \kappa^{-1} \overline{\text{sn}} \overline{\text{cn}}^2}{2(\kappa - 1)}, \quad (50)$$

$$\frac{\partial \overline{\text{dn}}}{\partial \kappa} + \frac{1}{\Omega_r} \frac{\partial \overline{\text{dn}}}{\partial u} = \frac{\overline{\text{dn}} \overline{\text{sn}}^2 - \overline{Z} \overline{\text{sn}} \overline{\text{cn}}}{2\kappa(\kappa - 1)}, \quad (51)$$

are expressed in terms of the Jacobi zeta function [2]

$$\overline{Z} \equiv Z(u|\kappa^{-1}) = \int_0^u \left(\text{dn}^2(s|\kappa^{-1}) - \frac{\overline{E}}{\overline{K}} \right) ds, \quad (52)$$

where $\overline{E} \equiv E(\kappa^{-1})$ and $\overline{K} \equiv K(\kappa^{-1})$. Using Eqs. (50)-(51), Eqs. (47)-(48) become

$$\frac{\partial \vartheta}{\partial J} = \frac{\omega_r}{2(\kappa - 1)} \left(\overline{Z} \overline{\text{dn}} - \kappa^{-1} \overline{\text{sn}} \overline{\text{cn}} \right), \quad (53)$$

$$\frac{\partial p}{\partial J} = \frac{\omega_r}{2\sqrt{\kappa}(\kappa - 1)} \left[\overline{\text{dn}} (\kappa - \overline{\text{cn}}^2) - \overline{Z} \overline{\text{sn}} \overline{\text{cn}} \right], \quad (54)$$

which again satisfies the canonical relation (27).

IV. GENERATING FUNCTION

Now that we have established the canonical nature of the phase-space transformation (26), we seek its generating function. First, we note that the canonical relation (27) can also be expressed as the differential two-form identity

$$dp \wedge d\vartheta \equiv dJ \wedge d\zeta, \quad (55)$$

where d denotes an exterior derivative (with $d^2 \equiv 0$). The differential canonical relation (55) therefore allows us to write the one-form $p d\vartheta$ in action-angle space as

$$p d\vartheta \equiv J d\zeta + dS, \quad (56)$$

where the function $S(\zeta, J)$ generates the canonical transformation (26), and the identity $d^2 S \equiv 0$ guarantees the canonical relation (55). Specifically, the generating function $S(\zeta, J)$ must satisfy the partial derivatives

$$\frac{\partial S}{\partial \zeta} \equiv p \frac{\partial \vartheta}{\partial \zeta} - J, \quad (57)$$

$$\frac{\partial S}{\partial J} \equiv p \frac{\partial \vartheta}{\partial J}, \quad (58)$$

for the libration and rotation cases, which we now investigate separately.

A. Libration Case

First, we consider the librating-pendulum case, represented by Eqs. (19)-(20), (31)-(32), and (40)-(41). Equation (57) for the libration case can be explicitly written as

$$\frac{\partial S_\ell}{\partial \zeta} = \frac{1}{\omega_\ell} \frac{\partial S_\ell}{\partial t} = \frac{4\kappa}{\omega_\ell} \text{cn}^2 - J_\ell, \quad (59)$$

where Eqs. (20) and (31) were used on the right side. Equation (59) can be integrated with respect to t to give

$$S_\ell(t, \kappa) = 4\kappa \int_0^t \text{cn}^2(s|\kappa) ds - J_\ell \zeta_\ell, \quad (60)$$

where we assumed that $S_\ell = 0$ at $t = 0$.

Next, we use the identity $\kappa \text{cn}^2 = \text{dn}^2 - (1 - \kappa)$, and the definition (38) for the Jacobi zeta function, to obtain

$$\begin{aligned} \int_0^t \kappa \text{cn}^2(s|\kappa) ds &= Z(t|\kappa) + [E - (1 - \kappa)K] \frac{t}{K} \\ &\equiv Z(t|\kappa) + \frac{1}{4} J_\ell \zeta_\ell, \end{aligned} \quad (61)$$

which, when inserted into Eq. (62), yields the final expression for the generating function

$$S_\ell(t, \kappa) = 4Z(t|\kappa). \quad (62)$$

Hence, we see that the Jacobi zeta function $Z(t|\kappa)$ generates the canonical transformation (26) for the librating-pendulum case. In the low-energy limit ($\kappa \ll 1$), Eq. (62) yields $S_\ell \simeq \kappa \sin 2\zeta_\ell$, where $J_\ell \simeq 2\kappa$ and $\zeta_\ell \simeq t$, which satisfies the conditions (57)-(58) with Eq. (22).

Lastly, the Jacobi zeta function (38) has the following partial derivatives

$$\frac{\partial Z}{\partial t} = \text{dn}^2 - \frac{E}{K}, \quad (63)$$

$$\frac{\partial Z}{\partial \kappa} + \frac{1}{\Omega_\ell} \frac{\partial Z}{\partial t} = \frac{\text{cn}}{2(1-\kappa)} (\text{dn sn} - Z \text{cn}), \quad (64)$$

where Ω_ℓ is defined in Eq. (30). These partial derivatives can be used to show that the generating function (62) satisfies the partial derivatives (57)-(58).

B. Rotation Case

Next, we consider the rotating-pendulum case, represented by Eqs. (23)-(24), (45)-(46), and (53)-(54). Equation (57) for the rotation case can be explicitly written as

$$\frac{\partial S_r}{\partial \zeta} = \frac{\sqrt{\kappa}}{\omega_r} \frac{\partial S_r}{\partial u} = \frac{4\kappa}{\omega_r} \overline{\text{dn}}^2 - J_r, \quad (65)$$

where Eqs. (24) and (45) were used on the right side. Equation (65) can be integrated with respect to u to give

$$S_r(u, \kappa) = 4\sqrt{\kappa} \int_0^u \text{dn}^2(s|\kappa^{-1}) ds - J_r \zeta_r, \quad (66)$$

where we assumed that $S_r = 0$ at $u = 0$.

Next, we use the definition (38) for the Jacobi zeta function to obtain

$$\begin{aligned} \int_0^u \text{dn}^2(s|\kappa^{-1}) ds &= Z(u|\kappa^{-1}) + \frac{\bar{E}}{K} u \\ &= Z(u|\kappa^{-1}) + \frac{J_r \zeta_r}{4\sqrt{\kappa}}, \end{aligned} \quad (67)$$

where we used Eqs. (14), (16), and (25). When this expression is inserted into Eq. (68), we obtain the final expression for the generating function for the rotation case

$$S_r(u, \kappa) = 4\sqrt{\kappa} Z(u|\kappa^{-1}). \quad (68)$$

Hence, like the libration case [Eq. (62)], the Jacobi zeta function plays a fundamental role in generating the canonical transformation (26) for the rotation case.

Lastly, the Jacobi zeta function (52) has the following partial derivatives

$$\frac{\partial \bar{Z}}{\partial u} = \overline{\text{dn}}^2 - \frac{\bar{E}}{\bar{K}}, \quad (69)$$

$$\frac{\partial \bar{Z}}{\partial \kappa} + \frac{1}{\Omega_r} \frac{\partial \bar{Z}}{\partial u} = \frac{\overline{\text{cn}}}{2\kappa(\kappa-1)} (\bar{Z} \overline{\text{cn}} - \overline{\text{dn sn}}), \quad (70)$$

where Ω_r is defined in Eq. (44). These partial derivatives can be used to show that the generating function (68) satisfies the partial derivatives (57)-(58).

V. SUMMARY

The problem of the motion of a pendulum represents a fundamental paradigm in mathematical physics. It is well known that its solution is intimately connected with the Jacobi elliptic functions, which represents an important class of mathematical functions that find applications in physics.

In the present paper, we have shown that yet another Jacobi elliptic function, the Jacobi zeta function $Z(t|\kappa)$, appears naturally in the canonical transformation (26) that defines the action-angle coordinates for the pendulum problem. Indeed, it is used to generate the canonical transformation for the libration motion (62) and the rotation motion (68) of the pendulum.

-
- [1] A. G. Greenhill, *The Applications of Elliptic Functions* (MacMillan and Co., London, 1892). The exact quote is “The determination of the (pendulum) motion introduces the *Elliptic Functions* in such an elementary and straightforward manner that we may take the elliptic functions as defined by pendulum motion, and begin the investigation of their use and theory by their application to this problem.”
 - [2] D. F. Lawden, *Elliptic Functions and Applications* (Springer-Verlag, New York, 1989).
 - [3] E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, 4th ed. (Dover, New York, 1937).
 - [4] L. D. Landau and E. M. Lifshitz, *Mechanics* (Elsevier, Amsterdam, 1976).
 - [5] A. J. Brizard, *An Introduction to Lagrangian Mechanics* (World Scientific, Singapore, 2008).
 - [6] A. J. Brizard, *Eur. J. Phys.* **30**, 729 (2009).
 - [7] B. C. Carlson, *Elliptic Integrals*, in *NIST Handbook of Mathematical Functions* (Cambridge University Press, 2010), chap. 19.
 - [8] W. P. Reinhart and P. L. Walker, *Jacobi Elliptic Functions*, in *NIST Handbook of Mathematical Functions* (Cambridge University Press, 2010), chap. 22.
 - [9] Here, we use the notation $\text{sn}(t|\kappa) \equiv \text{sn}(t, k)$, where $\kappa = k^2$.